

From perturbative calculations of the QCD static potential towards four-loop pole-running heavy quarks masses relation

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Abstract. The summary of the available semi-analytical results for the three-loop corrections to the QCD static potential and for the $\mathcal{O}(\alpha_s^4)$ contributions to the ratio of the running and pole heavy quark masses are presented. The procedure of the determination of the dependence of the four-loop contribution to the pole-running heavy quarks mass ratio on the number of quarks flavours, based on application of the least squares method is described. The necessity of clarifying the reason of discrepancy between the numerical uncertainties of the α_s^4 coefficients in the mass ratio, obtained by this mathematical method by the direct numerical calculations is emphasised.

1. Introduction.

It is known that in the Standard Model it is possible to introduce several definitions of heavy quarks masses. Rather applicable at present are the potential subtracted masses $m_{PS,q}(\mu^2)$ [1], the $\overline{\text{MS}}$ -scheme running masses $\overline{m}_q(\mu^2)$ and the scale-independent pole masses M_q . The relations between these definitions of heavy quarks masses were studied in the number of theoretical works on the subject (see e.g. [1], [2], [3]). Since these masses are extracted at present from different rather precise experimental data, e.g. from the spectroscopy of heavy hadrons and from the measurable cross-sections of the productions of heavy quarks in different observable processes, it is important to know the relations between M_q , $\overline{m}_q(\mu^2)$ and $m_{PS,q}(\mu^2)$ with high enough precision, which is gained by evaluation of high order perturbative corrections to the relations between pole and $\overline{\text{MS}}$ -scheme running heavy quarks masses and to the QCD static potential as well. This talk is devoted to the summary of the available results of the determination of three-loop corrections to the QCD static potential and to the $\mathcal{O}(\alpha_s^4)$ relations between different definitions of heavy quarks masses. Special attention is paid to the discussions of theoretical errors in the existing numerical calculations of the $\mathcal{O}(\alpha_s^4)$ contributions to the static potential [4],[5],[6] and to the ratio of the running and pole heavy quark masses [3]. The determination of the dependence of the this QCD correction on the number of lighter quarks flavours n_l by means of the ordinary least squares (OLS) mathematical method [7] is described.

2. The static potential in perturbative QCD.

The QCD static potential is strictly defined through the vacuum expectation value of the Wilson loop [8]:

$$V(r) = - \lim_{T \rightarrow \infty} \frac{1}{iT} \ln \frac{\langle 0 | \text{Tr} P e^{\frac{ig}{c} \oint_C dx^\mu A_\mu^a t^a} | 0 \rangle}{\langle 0 | \text{Tr} 1 | 0 \rangle} = \int \frac{d^3 \vec{q}}{(2\pi)^3} e^{i\vec{q}\vec{r}} \tilde{V}(\vec{q}^2) \quad (1)$$

where C is a closed rectangular contour, T and r are the time and three-dimensional space variables, P is the path-ordering operator, A_μ^a is a gluon field and t^a are the generators of Lie algebra of the $SU(N_c)$ group.

Perturbation theory expression for the Fourier transform $\tilde{V}(\vec{q}^2)$ of the static potential is known up to $\mathcal{O}(\alpha_s^4)$ -corrections:

$$\tilde{V}(q) = - \frac{4\pi C_F \alpha_s(\vec{q}^2)}{\vec{q}^2} \left(1 + a_1 a_s(\vec{q}^2) + a_2 a_s^2(\vec{q}^2) + \left(a_3 + \frac{\pi^2 C_A^3}{8} \ln \frac{\mu^2}{\vec{q}^2} \right) a_s^3(\vec{q}^2) \right) \quad (2)$$

where C_F and C_A are the Casimir operators and the QCD coupling constant $a_s(\vec{q}^2) = \alpha_s(\vec{q}^2)/\pi$ is defined in the $\overline{\text{MS}}$ -scheme. The coefficients a_1 and a_2 in Eq. (2) were obtained by evaluating in the analytical form the corresponding one and two-loops Feynman diagrams (see [9],[10] and [11], [12] respectively). The additional term $\pi^2 C_A^3/8$, correctly evaluated in [13], arises due to the infrared (IR) divergences, which begin to manifest themselves in the static potential at the three-loop level. However, in the concrete applications of the effective non-relativistic QCD these IR-divergent terms are cancelling out.

The three-loop contribution a_3 to Eq. (2) can be presented as

$$a_3 = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)} \quad (3)$$

where n_l are the number of quarks, which contribute to the QCD corrections to the static potential, responsible for the strong interactions of the heavy quark-antiquark pair with the flavour number $n_f = n_l + 1$. Note, that the masses of all lighter quarks with the flavour number n_l are usually neglected.

The n_l^3 and n_l^2 -dependent terms were evaluated analytically in [4] and have the following form

$$a_3^{(3)} = -\frac{125}{729} T_F^3, \quad a_3^{(2)} = \left(\frac{12541}{15552} + \frac{23}{12} \zeta(3) + \frac{\pi^4}{135} \right) C_A T_F^2 + \left(\frac{7001}{2592} - \frac{13}{6} \zeta(3) \right) C_F T_F^2 \quad (4)$$

Definite parts of the n_l -dependent coefficient $a_3^{(1)}$ and the overall expression for the constant term $a_3^{(0)}$ have not yet been computed analytically. The semi-analytical expression for $a_3^{(1)}$ was obtained in [4] and reads:

$$\begin{aligned} a_3^{(1)} = & -\frac{709.717}{64} C_A^2 T_F + \left(-\frac{71281}{10368} + \frac{33}{8} \zeta(3) + \frac{5}{4} \zeta(5) \right) C_A C_F T_F \\ & + \left(\frac{143}{288} + \frac{37}{24} \zeta(3) - \frac{5}{2} \zeta(5) \right) C_F^2 T_F - \frac{56.83(1)}{64} \frac{d_F^{abcd} d_F^{abcd}}{N_A} . \end{aligned} \quad (5)$$

where $d_F^{abcd} = \text{Tr}(t^a t^b t^c t^d)/6$ and $d_A^{abcd} = \text{Tr}(C^a C^b C^c C^d)/6$ is the total symmetric tensors, $(C^a)_{bc} = -if^{abc}$ are the generators of the adjoint representation of the Lie algebra of the $SU(N_c)$ group. Note that in the QED limit with $C_A=0$ the n_l -dependent terms, which are proportional to the powers of T_F in (4) and (5), are in agreement with the $\overline{\text{MS}}$ -scheme three-loop corrections to

the photon vacuum polarisation function in QED, presented previously in [19]. The inaccuracy of the numerical evaluation of the $C_A^2 T_F$ coefficient was not indicated in [4], since it is much smaller than the uncertainty of the $d_F^{abcd} d_A^{abcd}$ coefficient in the presented in (5) expression for the $a_3^{(1)}$ -term. The most precise calculation for $a_3^{(0)}$ was made in [5]. Its result

$$a_3^{(0)} = \frac{502.24(1)}{64} C_A^3 - \frac{136.39(12)}{64} \frac{d_F^{abcd} d_A^{abcd}}{N_A} \quad (6)$$

was published almost simultaneously with the following result of the independent calculation, performed in [6]:

$$a_3^{(0)} = \frac{502.22(12)}{64} C_A^3 - \frac{136.8(14)}{64} \frac{d_F^{abcd} d_A^{abcd}}{N_A} \quad (7)$$

Using the same computer code one of the authors of [6] improved recently the numerical precision of the second term in (7), which now reads¹:

$$a_3^{(0)} = \frac{502.22(12)}{64} C_A^3 - \frac{136.6(2)}{64} \frac{d_F^{abcd} d_A^{abcd}}{N_A} \quad (8)$$

Within the error bars the results (6), (7) and (8) have a common intersection region. The obtained in [5] contributions to $a_3^{(0)}$ have the smallest uncertainties. However, it is highly desirable to understand what is the reason of the differences in the numerical results of (6), (7) and (8).

Indeed, in the process of calculations the authors of [5] and [6] used the same theoretical methods. Let us mention the basic steps of these calculations. First, the diagrams, contributing to the 3-loop correction to static potential are generated. Then they are reduced to about 2000 3-loop integrals. In [5] these integrals were reduced to a relatively small number of the concrete master integrals using the integration-by-parts (IBP) method [14] and the FIRE algorithm [17], which is based in part on the Laporta algorithm [16]. Almost all master integrals were evaluated analytically apart of *three* of them, which were evaluated numerically. To check the results of these analytical and numerical calculations the program FIESTA [17], [18] was used. The obtained in [5] and [4] numerical results in the expressions for $a_3^{(0)}$ and $a_3^{(1)}$ terms are related to the results of the numerical evaluation of *three* mentioned above master integrals.

In the independently made work [6] the IBP method [14] and the Laporta algorithm were applied in the another way. To evaluate the master integrals the different from FIESTA computer program, written by the authors of [6] was used. Therefore, it is difficult to relate the numerical uncertainties of the results, given in (7) and (8) with the ones, presented in (6), which are determined by the numerical evaluation of the known *three* master integrals. In view of this it may be of real interest to consider the possibility of the detailed comparison of the FIESTA computer program with the one, created by the authors of [6].

Note, that in [20] using the maximal transcendentality hypothesis the guess on the analytical structure of *four* numerically known coefficients in the the expression for the $\mathcal{O}(a_s^3)$ correction to the static potential was made. More definitely, in [20] we proposed to consider the reliability of the existence of the following decomposition

$$709.717 = R_1 + R_2 \pi^2 + R_3 \pi^4 + R_4 \zeta(3) + R_5 \pi^2 \zeta(3) + R_6 \zeta(5) \quad (9)$$

$$502.24(1) = R_7 + R_8 \pi^2 + R_9 \pi^4 + R_{10} \zeta(3) + R_{11} \pi^2 \zeta(3) + R_{12} \zeta(5) \quad (10)$$

$$56.83(1) = R_{13} + R_{14} \pi^2 + R_{15} \pi^4 + R_{16} \zeta(3) \quad (11)$$

$$136.39(12) = R_{17} + R_{18} \pi^2 + R_{19} \pi^4 + R_{20} \zeta(3) \quad (12)$$

¹ We are grateful to Y. Sumino for informing us about this new unpublished result of his personal calculations.

where R_i are still unknown rational numbers. One should note that the concrete coefficients R_i in (9)-(12) may be zero. Considering carefully the analytical results of [4] in [20] we expressed the hope that R_5 , R_{11} , R_{12} and R_{18} may be really zero. The status of the discussed above hypothesis of [20] will be clarified when the remaining numerically calculated in [4] *three* master integrals will be evaluated analytically. The work on the solution of this calculation problem is now in progress ².

3. The relation between static potential and potential subtracted heavy quark masses.

The information on the QCD static potential and on the uncertainties of its perturbative calculations are important in the definitions of the heavy quarks masses and their values. This is possible to understand after consideration of introduced in [1] notion of heavy quark masses in the potential subtraction scheme. This definition is motivated by the studies of the behaviour of the cross-sections of the productions of heavy quarks near thresholds. It is based on the long-distance modifications of the perturbative static heavy quark potential $\tilde{V}(\vec{q}^2)$, oriented on the suppression of the IR renormalon contributions (for the consideration of the possible other power-suppressed non-perturbative contributions to the Coulomb-like potential for heavy quarks, which are not related to the IR renormalon effects, see e.g. [21]). In the static potential the IR effects can be eliminated by introducing boundary condition $|\vec{q}| > \mu_f$ to its Fourier transform, where the factorisation scale μ_f varies in the region $\Lambda_{QCD} < \mu_f < M_q v$, and v is the relative velocity of two heavy quarks, which is rather small near thresholds of their production. The equivalent way of the fixation of the factorisation scale is motivated by the uncertainty principle, which gives $\mu_f \sim 1/r \sim M_q v \sim M_q \alpha_s(M_q)$. The factorisation scale μ_f enter the definition of the subtracted potential $V(r, \mu_f)$ as:

$$V(r, \mu_f) = V(r) + 2\delta m_q(\mu_f) \quad . \quad (13)$$

The residual heavy quark mass $\delta m_q(\mu_f)$ is determined in the following way

$$\delta m_q(\mu_f) = -\frac{1}{2} \int_{|\vec{q}| < \mu_f} \frac{d^3 \vec{q}}{(2\pi)^3} \tilde{V}(\vec{q}^2) \quad (14)$$

where the perturbative expression for $\tilde{V}(\vec{q}^2)$ is presented above in (2). The defined by this way subtracted potential $V(r, \mu_f)$ does not contain long-distance contributions. As a consequence, in the calculations of the cross sections of heavy quark production in the near threshold region the pole mass M_q (which contain long-distance IR renormalon contributions in its relation to the $\overline{\text{MS}}$ -scheme running mass [22], [23]) should be changed to the potential subtracted quark mass $m_{\text{PS},q}(\mu_f)$, defined in [1] as

$$m_{\text{PS},q}(\mu_f) = M_q - \delta m_q(\mu_f) \quad . \quad (15)$$

In the r.h.s. of (15) the high loop perturbative corrections to $\delta m_q(\mu_f)$, which generate IR renormalon contributions, are cancelled exactly with long-distance contributions to the pole mass. Using now the expansion of the pole mass through the $\overline{\text{MS}}$ -scheme running mass

$$M_q = \overline{m}_q(\overline{m}_q^2) \left(1 + \sum_{i \geq 1} l_i a_s^i(\overline{m}_q^2)\right) \quad (16)$$

and substituting it into (15) one can eliminate the dependence on the factorisation scale μ_f and change it to the $\overline{\text{MS}}$ -scheme running mass $\overline{m}_q(\overline{m}_q^2)$. To finalise this section we note that

² We are grateful to V.A.Smirnov for this information.

it is possible to obtain the link between the pole heavy quark mass and the $\overline{\text{MS}}$ running mass through the subtracted-potential mass. Therefore the explicit expressions for the coefficients $a_3^{(0)}$ and $a_3^{(1)}$ in (3) can be related with the contributions to the $\mathcal{O}(a_s^4)$ coefficients in the the ratio of the running and pole heavy quark masses and may be used in future to check the results to be discussed in the next section.

4. The four-loop relation between $\overline{\text{MS}}$ running and pole heavy quark masses.

Perturbative relation between pole and $\overline{\text{MS}}$ running heavy quark masses contains the discovered in [22], [23] long-distance IR renormalon effects, which result from the asymptotic structure of the perturbative QCD series (for discussions of the asymptotic behaviour of the perturbative series in the renormalised quantum field theory models see [24]). However, to understand when the asymptotic nature of these expansions is starting to manifest itself and when the truncated perturbative series can be used it is necessary to study the concrete values of high-order corrections. To analyse these problems we consider the ratios of the defined in the $\overline{\text{MS}}$ -scheme running heavy quark masses and the pole heavy quark masses, namely

$$\frac{\overline{m}_q(M_q^2)}{M_q} = 1 + \sum_{i=1}^{\infty} z_m^{(i)} a_s^i(M_q^2) . \quad (17)$$

The coefficients $z_m^{(i)}$ can be expanded in powers of n_l . All coefficients $z_m^{(i)}$ with $1 \leq i \leq 3$ were evaluated analytically in the case of $\text{SU}(N_c)$ colour gauge group. The one-loop correction was found in [25], the two-loop contribution was analytically evaluated in [26], the three-loop term is known from the analytical and semi-analytical calculations, performed in [27] and [28] correspondingly. The coefficient $z_m^{(4)}$ can be presented in the analogous to equation (3) form:

$$z_m^{(4)} = z_m^{(40)} + z_m^{(41)} n_l + z_m^{(42)} n_l^2 + z_m^{(43)} n_l^3 \quad (18)$$

The terms $z_m^{(43)}$ and $z_m^{(42)}$ are known in the analytical form from the calculations of [29]. In the case of $\text{SU}_c(3)$ group of colour symmetry their expressions read:

$$z_m^{(43)} = \frac{42979}{1119744} + \frac{317\zeta_3}{2592} + \frac{89\pi^2}{3888} + \frac{71\pi^4}{25920} , \quad (19)$$

$$z_m^{(42)} = -\frac{32420681}{4478976} - \frac{40531\zeta_3}{5184} - \frac{63059\pi^2}{31104} - \frac{103\pi^2 \ln 2}{972} + \frac{11\pi^2 \ln^2 2}{243} - \frac{2\pi^2 \ln^3 2}{243} - \frac{5\pi^2 \zeta_3}{48} + \\ + \frac{241\zeta_5}{216} - \frac{30853\pi^4}{466560} - \frac{31\pi^4 \ln 2}{9720} + \frac{11 \ln^4 2}{486} - \frac{\ln^5 2}{405} + \frac{44}{81} \text{Li}_4\left(\frac{1}{2}\right) + \frac{8}{27} \text{Li}_5\left(\frac{1}{2}\right) \quad (20)$$

First two terms in (18) are not yet known analytically. However, the overall numerical expressions for $z_m^{(4)}$ at fixed n_l were calculated in [3] with the help of computer program FIESTA, created in [17], [18]. The obtained in [3] results read:

$$z_m^{(4)} \Big|_{n_l=3} = -1744.8 \pm 21.5, \quad z_m^{(4)} \Big|_{n_l=4} = -1267.0 \pm 21.5, \quad z_m^{(4)} \Big|_{n_l=5} = -859.96 \pm 21.5 \quad . \quad (21)$$

They were recently used by us in [7] to get the numerical expressions for the analytically unknown terms $z_m^{(40)}$ and $z_m^{(41)}$ with the help of rigorous optimal least squares (OLS) mathematical method. Bellow we summarise definite steps of this work and discuss its main results. To get the values of *two* unknown parameters from the results of (21) we use the following overdetermined system of *three* linear equations:

$$z_m^{(40)} + 3z_m^{(41)} = -1371.77 , \quad z_m^{(40)} + 4z_m^{(41)} = -614.68 , \quad z_m^{(40)} + 5z_m^{(41)} = 142.32 \quad (22)$$

To solve this system by means of the OLS method the following function is introduced

$$\Phi(z_m^{(40)}, z_m^{(41)}) = \sum_{k=1}^3 \Delta_{l_k}^2 = \sum_{k=1}^3 (z_m^{(40)} + z_m^{(41)} n_{l_k} - y_{l_k})^2$$

where Δ_{l_k} are the squared deviations, $1 \leq k \leq 3$ labels the concrete equations in the system (22) and the expressions for y_{l_k} are fixed by the numbers on the r.h.s. of each equation from this system. Its solutions $(z_m^{(40)}, z_m^{(41)})$ are determined by the following requirements:

$$\frac{\partial \Phi}{\partial z_m^{(40)}} = 0, \quad \frac{\partial \Phi}{\partial z_m^{(41)}} = 0. \quad (23)$$

It is possible to show that their solution always exists and is unique due to the fact that the rank of the matrix, composed from the l.h.s. of the system, is equal to two and coincides with the number of unknowns parameters. This method allows to determine theoretical uncertainties of the solutions of Eq. (23) of the parameters. Theoretical errors were defined in [7] as

$$\Delta z_m^{(40)} = \frac{\sqrt{\sum_{k=1}^3 n_{l_k}^2}}{\sqrt{3 \sum_{k=1}^3 n_{l_k}^2 - \left(\sum_{k=1}^3 n_{l_k}\right)^2}} \Delta y_l, \quad \Delta z_m^{(41)} = \frac{\sqrt{3} \Delta y_l}{\sqrt{3 \sum_{k=1}^3 n_{l_k}^2 - \left(\sum_{k=1}^3 n_{l_k}\right)^2}} \quad (24)$$

where $\Delta y_l = \sigma = 21.5$ are the given in [3] uncertainties of the results of (21). Using equations (23), supplemented by (24), in [3] we found the following expressions for two parameters we are interested in :

$$z_m^{(40)}(M_q^2) = -3642.9 \pm 62.0, \quad z_m^{(41)}(M_q^2) = 757.05 \pm 15.2. \quad (25)$$

Similar expressions were also obtained in [30] with the help of the *fitting procedure* of the numerical results of [3] (see Eq. (21)), supplemented by the calculated in [29] analytical expressions for $z_m^{(43)}$ and $z_m^{(42)}$ and by renormalon-based large β_0 -representation of the n_l -dependence for $z_m^{(4)}$, fixed in [31], [22]. This additional theoretical input was not used by us. However, our results from Eq. (25) agree very well with the numbers

$$z_m^{(40)}(M_q^2) = -3643 \pm 21.5, \quad z_m^{(41)}(M_q^2) = 757 \pm 21.5, \quad (26)$$

which follow from the ones obtained in [30] for the $\mathcal{O}(a_s^4)$ contributions to the inverse ratio M_q/\overline{m}_q , where the running mass was normalised at another scale $\mu = \overline{m}_q$. The essential difference is that in [30] theoretical errors of the outcomes of their fits were not analysed separately but fixed by the errors of the the numerical results, obtained in [3].

5. Discussions of the theoretical errors.

In our work of [7] we raised the question on the reason of the coincidence of the numerical errors in the presented in [3] results of the numerical calculations of the $\mathcal{O}(a_s^4)$ corrections to the relations between running and pole heavy masses of c , b and t -quarks (see Eq. (21)). The proposed in [7] explanation of the flavour-independence of the numerical error is that this error corresponds to the uncertainty of the n_l -independent contribution to (18), namely of the coefficient $z_m^{(40)}$. If this explanation is correct, then the error of the numerical calculations of the coefficient $z_m^{(41)}$ should be negligibly small. Therefore, the given in [30] errors are not absolutely correct. Indeed, they do not satisfy the proposed in [7] explanation of the “paradox of the coincidence

of the numerical errors". The determined in [7] OLS method error of $z_m^{(40)}$ -term really dominate over the uncertainty of $z_m^{(41)}$ (see (25)) and it is 3 times larger than the numerical error of this coefficient, which follows from the results of [3]. In spite of this positive fact we understand that the errors obtained within the OLS method may be overestimated (in order to get more definite estimates it is necessary to have more than three equations). However, even these results indicate that it may be useful to clarify in more details how the numerical errors of the results of [3] were obtained. This problem is really important. Indeed, the difference in the numerical errors discussed above leads to different theoretical estimates of the uncertainties of the top-quark pole mass value (in the case of applications of the results of [3] this error is 0.005 GeV, while if the results of [7] are used the error is 4 times larger, namely 0.023 GeV). The analytical evaluation of the coefficients $z_m^{(40)}$ and $z_m^{(41)}$ will remove this problem. It is highly desirable to get their analytical expressions. These calculations are realistic and already started in [32] from the creation of the special computer program.

6. Acknowledgements

We are grateful to V.A. Smirnov for discussions of the definite subjects discussed in this talk. One of us (ALK) would like to thank the Organisers of ACAT-2016 Workshop in Valparaiso, Chile (18-22 January 2016) for the invitation and partial financial support. This work of ALK was supported by Russian Science Foundation, grant No. 14-22-00161.

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